Exercise 1.1.2

If $\lim_{n\to\infty} \frac{b_n}{a_n} = K$, a constant with $0 < K < \infty$, show that $\sum_n b_n$ converges or diverges with $\sum a_n$.

Hint. If $\sum a_n$ converges, rescale b_n to $b_n' = \dfrac{b_n}{2K}$. If $\sum_n a_n$ diverges, rescale to $b_n'' = \dfrac{2b_n}{K}$.

Solution

Suppose that

$$\lim_{n\to\infty} \frac{b_n}{a_n} = K,$$

where $0 < K < \infty$. According to the precise definition of an infinite limit, for all $\epsilon > 0$ there exists a number N such that

$$n > N \quad \Rightarrow \quad \left|\frac{b_n}{a_n} - K\right| < \epsilon.$$

Rewrite the consequent.

$$-\epsilon < \frac{b_n}{a_n} - K < \epsilon$$

$$K - \epsilon < \frac{b_n}{a_n} < K + \epsilon$$

$$(K - \epsilon)a_n < b_n < (K + \epsilon)a_n$$

If $\sum_n a_n$ converges, then $\sum_n b_n$ converges as well by the direct comparison test.

$$\sum_n b_n < \sum_n (K + \epsilon)a_n = (K + \epsilon) \underbrace{\sum_n a_n}_{\text{converges}} \qquad \Rightarrow \qquad \sum_n b_n \text{ converges}$$

If $\sum_n a_n$ diverges, then $\sum_n b_n$ diverges as well by the direct comparison test.

$$\sum_n (K - \epsilon)a_n = (K - \epsilon) \underbrace{\sum_n a_n}_{\text{diverges}} < \sum_n b_n \qquad \Rightarrow \qquad \sum_n b_n \text{ diverges}$$